Systems of Rays Normal to a Surface.

By W. C. L. Gorton.

The following article is intended as a supplement to §7 of my paper in this Journal, Vol. X, p. 347. It treats primarily of systems of rays originally passing through a point, which we shall call the focus. The results are not new, and are only given as illustrations of the method employed and the ready applicability of quaternions to such problems. We shall use the laws of reflection and refraction determined by experiment, viz. I. The incident and reflected rays are in the same plane with the normal to the reflecting surface and make equal angles with it; II. The incident and refracted rays are in the same plane with the normal to the refracting surface, and the sines of the angles which they make with it bear a constant ratio to each other. Let $\rho = \sigma + x\tau$ be the equation of a system of rays where $\sigma = f(t, u)$, $\tau = \psi(t, u)$ and $T\tau = 1$. If for any value of x such as $x = \phi(t, u)$ they be normal to a surface, we must have

$$S\tau \partial_t \rho = S\tau \partial_u \rho = 0,$$

where

$$\partial_t \rho = \partial_t \sigma + x \partial_t \tau + \tau \partial_t x$$

and

$$\partial_u \rho = \partial_u \sigma + x \partial_u \tau + \tau \partial_u x,$$

$$\therefore \ S\tau \partial_t \sigma + x S\tau \partial_t \tau + \tau^2 \partial_t x = S\tau \partial_t \sigma - \partial_t x = 0,$$
$$S\tau \partial_u \sigma + x S\tau \partial_u \tau + \tau^2 \partial_u x = S\tau \partial_u \sigma - \partial_u x = 0,$$

since $S\tau \partial_t \tau = S\tau \partial_u \tau = 0$ and $\tau^2 = -1$, since

$$\frac{d^2 x}{dt\, du} = \frac{d^2 x}{du\, dt}$$

we must have

$$\frac{\partial}{\partial u} S\tau \partial_t \sigma - \frac{\partial}{\partial t} S\tau \partial_u \sigma = 0,$$

and therefore we obtain as a necessary and sufficient condition

$$S\partial_t\tau\partial_u\sigma - S\partial_u\tau\partial_t\sigma = 0.$$

Let us now consider rays emanating from a point which have been reflected by some surface.

Let $\rho = x\sigma + y\tau$ be the equation of the reflected system where $\rho = x\sigma$ is the equation of the reflecting surface and $T\sigma = T\tau = 1$. Calling ν the normal to the surface $\rho = x\sigma$, since σ and τ are unit vectors, and ν bisects the angle between them, we have

$$\nu \| \tau - \sigma,$$
$$\therefore S(\tau - \sigma)\partial_t x\sigma = 0,$$
$$S(\tau - \sigma)\partial_u x\sigma = 0,$$

which give us

$$S\tau\partial_t x\sigma + \partial_t x = 0$$

and

$$S\tau\partial_u x\sigma + \partial_u x = 0.$$

Differentiating the first with respect to t and the second with respect to u and subtracting, we have

$$S\partial_t\tau\partial_u x\sigma - S\partial_u\tau\partial_t x\sigma = 0,$$

or the reflected system is normal to some surface. Therefore, in order that a system of rays may be brought to a focus by reflection they must be normal to a surface.

Let $\rho = \sigma + x\tau_1$ be the equation of a normal system of rays; then if the rays be reflected by the surface $\rho = \sigma + x_1\tau_1$ where $x_1 = f(t, u)$, we shall have as the equation of the reflected system,

$$\rho = \sigma + x_1\tau_1 + x\tau,$$

where

$$T\tau_1 = T\tau = 1.$$

Calling ν_1 the normal to the reflecting surface, we have

$$\nu_1 \| \tau_1 - \tau,$$
$$\therefore S(\tau_1 - \tau)\partial_t(\sigma + x\tau) = 0,$$
$$S(\tau_1 - \tau)\partial_u(\sigma + x\tau) = 0.$$

Treating these as above, we have

$$S\partial_t\tau_1\partial_u(\sigma + x\tau) - S\partial_u\tau\partial_t(\sigma + x\tau) = S\partial_t\tau\partial_u(\sigma + x\tau) - S\partial_u\tau\partial_t(\sigma + x\tau)$$
$$= S\partial_t\tau\partial_u\sigma - S\partial_u\tau\partial_t\sigma$$
$$= 0,$$

since the system $\rho = \sigma + x\tau$ was normal to a surface. Therefore, a system of rays which is normal to a surface is still normal to some surface after any number of reflections.

Let $\rho = x\sigma$ be the equation of a system of rays emanating from a point. After reflection by the surface $\rho = x_1\sigma$ their equation will be of the form $\rho = x_1\sigma + y\tau$, and let $\rho = x_1\sigma + y_1\tau$ be the equation of one of the surfaces to which the reflected system is normal. Then since $\tau - \sigma$ is normal to the surface $\rho = x_1\sigma$ and τ to the surface $\rho = x_1\sigma + y_1\tau$, we have

$$S(\sigma - \tau)\,\partial_t x_1\sigma = 0,$$
$$S\tau\partial_t (x_1\sigma + y_1\tau) = 0.$$

Adding these equations we obtain

$$S\sigma\partial_t x_1\sigma + S\tau\partial_t y_1\tau = 0,$$
or
$$\partial_t x + \partial_t y = 0.$$

In the same way we can prove

$$\partial_u x + \partial_u y = 0.$$

Therefore the distance along any ray from the focus to the normal surface is independent of the ray. This can readily be generalized as follows:

Let
$$\rho = x_1\tau_1 + x_2\tau_2 + \cdots + x_n\tau_n + x\tau$$

be the equation of one of the surfaces to which the rays are normal after n reflections. Then, on account of the law of reflection, we have

$$S(\tau_2 - \tau_1)\,\partial_t x_1\tau_1 = 0,$$
$$S(\tau_3 - \tau_2)\,\partial_t (x_1\tau_1 + x_2\tau_2) = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S(\tau - \tau_n)\,\partial_t (x_1\tau_1 + x_2\tau_2 + \cdots + x_n\tau_n) = 0.$$

Adding these equations, we obtain

$$S\tau_1\partial_t x_1\tau_1 + \cdots + S\tau_n\partial_t x_n\tau_n - S\tau\partial_t (x_1\tau_1 + \cdots + x_n\tau_n) = 0,$$
but
$$S\tau\partial_t (x_1\tau_1 + \cdots + x_n\tau_n) = -S\tau\partial_t x\tau,$$
$$\therefore S\tau_1\partial_t x_1\tau_1 + \cdots + S\tau_n\partial_t x_n\tau_n + S\tau\partial_t x\tau = 0.$$

Expanding and remembering that $T\tau_1 = \ldots = T\tau_n = T\tau = 1$, we have

$$\partial_t x_1 + \ldots + \partial_t x_n + \partial_t x = 0 .$$

In the same way we can prove

$$\partial_u x_1 + \ldots + \partial_u x_n + \partial_u x = 0 .$$

Therefore, after any number of reflections the distance along any ray from the focus to a normal surface is independent of the ray. These results can readily be extended to the case of refraction with such differences as the difference in the law of refraction introduces.

Let $\rho = x_1\tau_1 + x\tau$ be the equation of a system of rays which emanating from a point have been refracted at the surface $\rho = x_1\tau_1$. Let n_1 and n be the indices of refraction of the two media and $T\tau = T\tau_1 = 1$. By the law of refraction, calling ν the normal to the refracting surface, we have

$$\nu \,\| \, n\tau - n_1\tau_1 ,$$
$$\therefore \; S\left(n\tau - n_1\tau_1\right) \partial_t x_1\tau_1 \;= 0 ,$$
$$S\left(n\tau - n_1\tau_1\right) \partial_u x_1\tau_1 = 0 ;$$

expanding
$$nS\tau\partial_t x_1\tau_1 = - n_1\partial_t x_1 ,$$
$$nS\tau\partial_u x_1\tau_1 = - n_1\partial_u x_1 ,$$
$$\therefore \; S\partial_t\tau\partial \; x_1\tau_1 - S\partial_u\tau\partial_t x_1\tau_1 = 0 ,$$

and the refracted rays are normal to some surface.

Let $\rho = \sigma + x\tau$ be the equation of any normal system of rays, and let $\rho = \sigma + x\tau + x_1\tau_1$ be the equation of the system after refraction at the surface $\rho = \sigma + x\tau$.

Let m be the index of refraction of the first medium and n that of the second, then

$$S\left(m\tau - n\tau_1\right) \partial_t \left(\sigma + x\tau\right) = 0 ,$$
$$S\left(m\tau - n\tau_1\right) \partial_u \left(\sigma + x\tau\right) = 0 ,$$
$$nS\tau_1\partial_t \left(\sigma + x\tau\right) = m\partial_t x - mS\tau\partial_t\sigma ,$$
$$nS\tau_1\partial_u \left(\sigma + x\tau\right) = m\partial_u x - mS\tau\partial_u\sigma .$$

Differentiating the first with respect to u and the second with respect to t and subtracting,

$$m\left(S\partial_u\tau_1\partial_t(\sigma + x\tau) - S\partial_t\tau\partial_u(\sigma + x\tau)\right) = m\left(S\partial_t\tau\partial_u\sigma - S\partial_u\tau\partial_t\sigma\right)$$
$$= 0;$$

therefore we have the general theorem that a system of rays normal to a surface are still normal to some surface after any number of reflections and refractions.

Let us consider the equation $\rho = x_1\tau_1 + x\tau$ of a surface to which the refracted rays, originally emanating from a point, are normal. Calling the indices of refraction n_1 and n, we have

$$S(n_1\tau_1 - n\tau)\,\partial_t x_1\tau_1 = 0,$$
$$Sn\tau\partial_t(x_1\tau_1 + x\tau) = 0.$$

Adding, we have $\qquad n_1 S\tau_1\partial_t x_1\tau_1 + nS\tau\partial_t x\tau = 0,$

or $\qquad n_1\partial_t x_1 + n\partial_t x = 0.$

In the same way we can prove that

$$n_1\partial_u x_1 + n\partial_u x = 0.$$

This can be easily generalized as follows:

Let $\rho = x_1\tau_1 + x_2\tau_2 + \ldots + x_n\tau_n + x\tau$ be the equation of a surface to which the rays are normal after n refractions. Let $n_1, n_2, \ldots n_n$, n be the indices of refraction, and $T\tau_1 = T\tau_2 = \ldots = T\tau_n = T\tau = 1$. We have, by the laws of refraction,

$$S(n_1\tau_1 - n_2\tau_2)\,\partial_t x_1\tau_1 \qquad = 0,$$
$$S(n_2\tau_2 - n_3\tau_3)\,\partial_t(x_1\tau_1 + x_2\tau_2) = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S(n_n\tau_n - n\tau)\,\partial_t(x_1\tau_1 + \ldots + x_n\tau_n) = 0.$$

Since, then, rays are normal to the above surface, we have

$$S\tau\partial_t(x_1\tau_1 + \ldots + x_n\tau_n + x\tau) = 0,$$

and therefore $\qquad nS\tau\partial_t(x_1\tau_1 + \ldots + x_n\tau_n) = -nS\tau\partial_t x\tau.$

23

Adding the above equations and making use of this relation, we have

$$n_1 S \tau_1 \partial_t x_1 \tau_1 + \ldots + n_n S \tau_n \partial_t x_n \tau_n + n S \tau \partial_t x \tau = 0,$$

or
$$n_1 \partial_t x_1 + \ldots + n_n \partial_t x_n + n \partial_t x \qquad = 0.$$

In the same way we can prove

$$n_1 \partial_u x_1 + \ldots + n_n \partial_u x_n + n \partial_u x = 0.$$

Therefore, if we consider the path of any ray from the focus to the normal surface, we have the theorem that sum of the lengths of the path in each medium multiplied by the corresponding index of refraction is independent of the ray.